Counting Subsets

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Counting subsets of a finite set

Theorem: A finite set with n elements has 2^n subsets.

$\{1, 2\}$ has 4 subsets:

- the empty set $\emptyset$,
- the one element sets $\{1\}$ and $\{2\}$,
- the two element set $\{1, 2\}$.

$\{a, b, c\}$

$\emptyset$ 1

$\{a\}$ $\{b\}$ $\{c\}$ 3

$\{a, b\}$ $\{b, c\}$ $\{a, c\}$ 3

$\{a, b, c\}$ 1

$\{a, b, c, d\}$ / $\emptyset$

4 $\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$

?

4 $\{a, b, c\}$ $\{a, b, d\}$ $\{a, c, d\}$
$\{b, c, d\}$

1 $\{a, b, c, d\}$

Counting subsets

The book gives one explanation for why this is true on page 13. We will give a slighly different one.

Counting subsets

Suppose we have a finite set A with n elements. We will list the elements as $a_1, a_2, \ldots, a_n$.

$$A = \{a_1, a_2, \ldots, a_n\}.$$

Here, we've decided to put the elements of A in order, but it doesn't matter what order you use.

Counting subsets

A subset B of A is determined by going through the elements of A and marking each element as either "in" our "out" of the subset. So we can describe a subset of A by giving a list

$$I, I, O, I, O, \ldots, I$$

where we have an I if that element is in the subset, or an O if it isn't.

Counting example

Suppose $A = \{-1, 4, 7, 8\}$. We put the elements of A in that order, so $a_1 = -1$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 8$. Let $B = \{-1, 7\}$ so that $B \subseteq A$.

Then B corresponds to the list

$$I, O, I, O$$

since -1 is IN B, 4 is OUT of B, 7 is IN B, and 8 is OUT of B.

The list O, I, O, O corresponds to the subset $\{4\}$ since only 4 is IN this set.

Counting subsets

Theorem: The number of subsets of a set A with n elements is the same as the number of ordered sequences of I and O of length n, and this number is 2^n.

Proof: Let $S = \{I, O\}$. We've seen above how a sequence of I and O correspond to a subset. The set of sequences of I and O of length n is exactly S^n. By our earlier counting result, $|S^n| = |S|^n = 2^n$.

$S = \{I, O\}$

$|A \times B| = |A||B|$

$|A^n| = |A|^n$

$\underbrace{S \times S \times S \cdots \times S}_{n \text{ times}}$

A m els

B n els

$A \times B$