

Counting Subsets

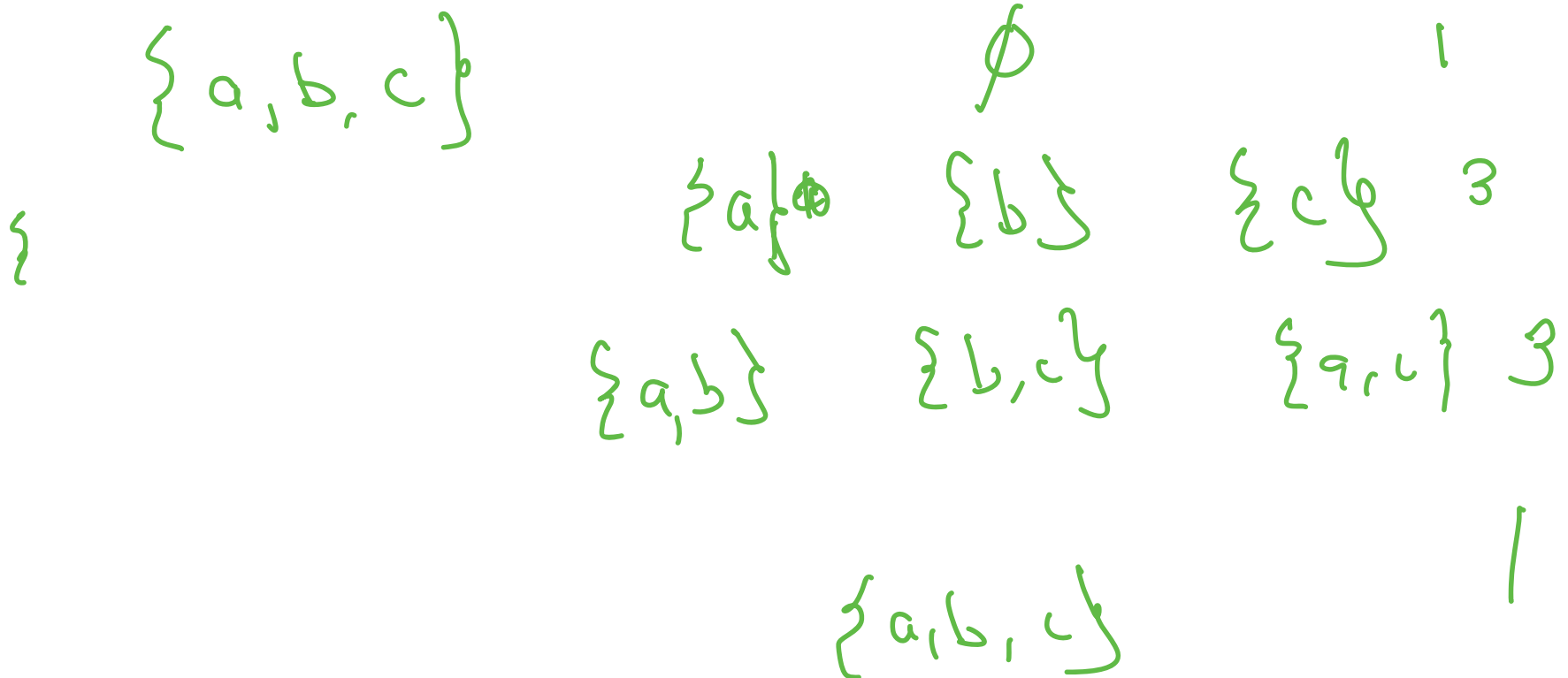
Counting Subsets

Counting subsets of a finite set

Theorem: A finite set with n elements has 2^n subsets.

$\{1, 2\}$ has 4 subsets:

- ▶ the empty set $\emptyset$,
- ▶ the one element sets $\{1\}$ and $\{2\}$,
- ▶ the two element set $\{1, 2\}$.



$\{a, b, c, d\},$

$\emptyset$

$\{$

$a\}$

$\{b\}$

$\{c\}$

$\{d\}$

$?$

$\{$

$a, b, c\}$

$\{c, b, d\}$

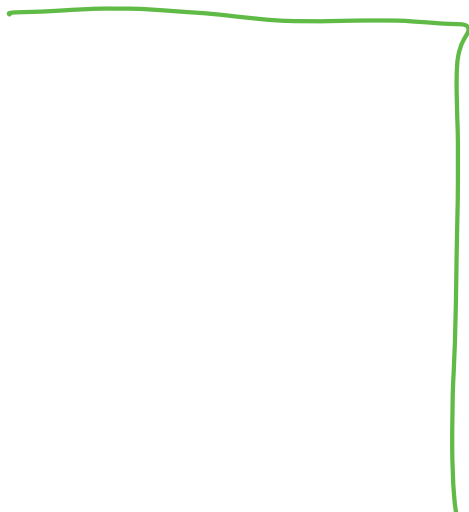
~~$\{a, b, c\}$~~

$\{a, c, d\}$

$\{b, c, d\}$

$|$

$\{a, b, c, d\}$



Counting subsets

The book gives one explanation for why this is true on page 13. We will give a slightly different one.

Counting subsets

Suppose we have a finite set A with n elements. We will list the elements as $a_1, a_2, \dots, a_n$.


$$A = \{a_1, a_2, \dots, a_n\}.$$

Here, we've decided to put the elements of A in order, but it doesn't matter what order you use.

Counting subsets

A subset B of A is determined by going through the elements of A and marking each element as either “in” or “out” of the subset. So we can describe a subset of A by giving a list

$$I, I, O, I, O, \dots, I$$

where we have an I if that element is in the subset, or an O if it isn't.

Counting example

Suppose $A = \{-1, 4, 7, 8\}$. We put the elements of A in that order, so $a_1 = -1$, $a_2 = 4$, $a_3 = 7$, and $a_4 = 8$. Let $B = \{-1, 7\}$ so that $B \subseteq A$.

Then B corresponds to the list

I, O, I, O

since -1 is IN B , 4 is OUT of B , 7 is IN B , and 8 is OUT of B .

The list *O, I, O, O* corresponds to the subset $\{4\}$ since only 4 is IN this set.

Counting subsets

Theorem: The number of subsets of a set A with n elements is the same as the number of ordered sequences of I and O of length n , and this number is 2^n .

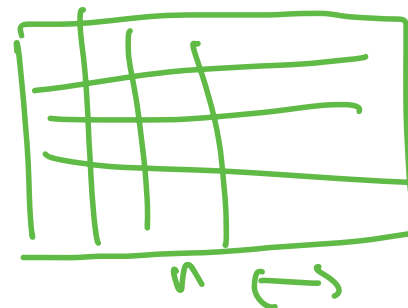
Proof: Let $S = \{I, O\}$. We've seen above how a sequence of I and O correspond to a subset. The set of sequences of I and O of length n is exactly S^n . By our earlier counting result, $\underbrace{|S^n|}_{=} = \underbrace{|S|^n}_{=} = 2^n$.

$$S = \{I, O\}$$

$$\underbrace{S \times S \times S \cdots \times S}_{n \text{ times}}$$

$$\begin{array}{l} A \text{ } m \text{ el } B \\ B \text{ } n \text{ el } A \end{array}$$

$$A \times B$$



$$\begin{aligned} |A \times B| &= |A| |B| \\ |A^n| &= |A|^n \end{aligned}$$